CLASSIFICATION AND CONSTRUCTION OF QUASISIMPLE LIE ALGEBRAS

Raphaël HØEGH-KROHN(*)   Bruno TORRESANI(**)

Abstract : We study a class of (possibly infinite-dimensional) Lie algebras, called the Quasisimple Lie algebras (QSLA's), and generalizing semisimple and affine Kac-Moody Lie algebras. They are characterized by the existence of a finite-dimensional Cartan subalgebra, a non-degenerate symmetric ad-invariant Killing form, and nilpotent rootspaces attached to non-isotropic roots. We are then able to derive a classification theorem for the possible quasisimple root systems; moreover, we construct explicit realizations of some of them as current algebras, generalizing the affine loop algebras.

(*) Universitetet i Oslo, Matematisk Institutt. Blindern Oslo 3, Oslo, Norway

(**) Allocataire du M.R.T.

(+) Laboratoire propre du CNRS - LP 7061

I. INTRODUCTION:

Infinite-dimensional Lie algebras and their representations have been shown to be a very exciting and powerful tool for the investigation of many apparently disconnected fields in mathematics and mathematical physics, like combinatorial identities [2], [3], non-linear partial differential equations of "soliton type" [4], dual resonance models and string theory [5], and anomalies in quantum gauge theories [6] for instance.

In fact, it is hoped that they allow a better understanding of the remarkable interrelations between these different fields. Among the class of infinite-dimensional Lie algebras, there exists a remarkable subclass, i.e. the affine Kac-Moody Lie algebras.

Introduced in 1968 by Kac [7] and Moody [8] separately, they generalized Serre's reconstruction theorem (concerning the semisimple case), and have been the starting point of considerably many studies (see ref. [1] and references therein for a survey). In particular, in connection with string theories, explicit realizations of the simply laced affine Lie algebras (i.e. $\mathbf{A}_n^{(1)}$, $\mathbf{D}_n^{(1)}$, $\mathbf{E}_n^{(1)}$), together with their "basic representation" (see [1] for instance) have been worked out [5], [9], using as a basic tool the vertex operator, describing the emission of a "tachyon" by a bosonic string [10].

In this work, we are interested in higher-dimensional generalizations of affine Lie algebras ; the aim of this paper is in fact to describe a class of (possibly infinite-dimensional) Lie algebras, including as particular cases the semisimple and the affine cases, and to derive their general properties. We then choose to study Lie algebras, called the quasisimple Lie algebras (Q.S.L.A's) characterized by properties which appear to be fairly natural and not so much restrictive :

- finite-dimensional Cartan subalgebra
- non-degenerate ad-invariant Killing form
- discrete root system
- ad-nilpotency of the rootspaces attached to non-isotropic roots.

We are then able to derive a classification theorem for the possible quasisimple root systems, and moreover to construct an explicit realization of some QSLA's as "current algebras", generalizing the loop algebras. This approach appears to be very interesting for the study of quantum gauge theories, in the sense that the current algebra realizations, of the form $P(T^{\nu},\mathbf{g}_0)$, T^{ν} being the ν-dimensional torus and $\mathbf{g}_0$ a semisimple Lie algebra, provide us with a nice tool to investigate the infinitesimal unitary highest weight representations of local gauge transformations groups [11], [15].

The paper is organized as follows : in section **II**, we derive some general properties of the QSLA's, and conclude with a classification theorem of all possible irreducible "elliptic" quasisiple root systems ; in section **III**, we build explicit realizations of some QSLA's as current algebras ; section **IV** is devoted to the conclusions.

II. DEFINITIONS AND CLASSIFICATION:

1. Definition 1:

Let $\mathbf{g}$ be a complex Lie algebra ; $\mathbf{g}$ is said to be quasisimple if :

(Q.S.L.A. 1) : $\mathbf{g}$ is provided with a non degenerate invariant symmetric bilinear form, called the Killing form, and denoted by : $<,>$

(Q.S.L.A.2) : $\mathbf{g}$ possesses a Cartan subalgebra $\mathbf{h}$, such that

- $\mathbf{h}$ is diagonalizable
- $\mathbf{h}$ is finite-dimensional
- ad($\mathbf{h}$) has discrete spectrum

With respect to ad($\mathbf{h}$), $\mathbf{g}$ possesses a rootspace decomposition :

$$\mathbf{g} = \mathbf{h} \oplus \sum_{\alpha \in \mathbf{R}}^{\oplus} \mathbf{g}_\alpha$$

where $\mathbf{R} = \mathrm{Sp}\,[\mathrm{ad}(\mathbf{h})]$ (1)

and $\mathbf{g}_\alpha = \{x \in \mathbf{g} \text{ s.t. } \mathrm{ad}(h).x = \alpha(h)x \quad \forall\, h \in \mathbf{h}\}$

Moreover, the Killing form induces a bilinear form on the dual $\mathbf{h'}$ of $\mathbf{h}$, and the last assumption is :

(Q.S.L.A.3) : for any non-isotropic root α (i.e. $<\alpha,\alpha> \neq 0$), ad($\mathbf{g}_\alpha$) is nilpotent.

<u>Remark</u> : It is easily seen that semisimple and Kac-Moody affine Lie algebras are quasisimple.

2. General properties :

In this subsection, we enumerate a sequence of general properties ; the proofs are fairly standard and will not be mentioned here.

Theorem 1 :

i) For any pair α,β of roots of $\mathbf{g}$, if $\alpha + \beta \neq 0$, then $<\mathbf{g}_\alpha, \mathbf{g}_\beta> = 0$; on the other hand, $<,>$ is non-degenerate on $\mathbf{h}\times\mathbf{h}$ and $\mathbf{g}_\alpha\times\mathbf{g}_{-\alpha}$, for any $\alpha \in \mathbf{R}$

ii) $-\mathbf{R} = \mathbf{R}$

iii) For any pair α,β of roots : $[\mathbf{g}_\alpha,\mathbf{g}_\alpha] \subset \mathbf{g}_{\alpha+\beta}$

If $\alpha + \beta$ is not a root : $\mathbf{g}_{\alpha+\beta} = \{0\}$

iv) For any root α, $[\mathbf{g}_\alpha,\mathbf{g}_{-\alpha}]$ is the subspace of h generated by h_α, h_α being the canonical image of α under the identification of **h** and **h'**.

<u>Remark :</u> The non-degeneracy of the Killing form allows the identification of **h** with **h'**, in the usual way :

for any $\gamma \in \mathbf{h}'$, define h_γ by :

$$<h_\gamma, h> = \gamma(h) \quad , \ \forall\, h \in \mathbf{h}.$$

This allows to carry the Killing form on **h'** :

$\forall\, \alpha,\beta \in \mathbf{h}' : <\alpha,\beta> = <h_\alpha,h_\beta>$.

Definition 2 :

A root α of **g** is called an isotropic root if $<\alpha,\alpha> = 0$

3. The non-isotropic roots :

The non-isotropic roots have exactly the same behaviour as the roots of a semisimple Lie algebra (see [12], [13] for instance). Then, we will state, without proof, the following recapitulation theorem :

Theorem 3 :

Let **g** be a quasisimple Lie algebra, and let **R** be its root system ; let α be a non-isotropic root in **R**.

i) $\dim \mathbf{g}_\alpha = 1$

ii) $k\alpha$ is a root if and only if $k = \pm 1$.

iii) for any root β , $2 <\alpha,\beta>/<\alpha,\alpha> \in \mathbb{Z}$

iv) for any root β, $w_\alpha.\beta = \beta - [2<\alpha,\beta>/<\alpha,\alpha>]\,\alpha$ is a root too.

then $w_\alpha.\mathbf{R} = \mathbf{R}$

v) for any root β, the following statement is true : there exist two non-negative integral numbers n_+ , n_- such that $\beta + n\,\alpha \in \mathbf{R}$ if and only if $-n_- \leqslant n \leqslant n_+$. Moreover, we have $n_- - n_+ = 2 <\alpha,\beta>/<\alpha,\alpha>$.

In the semisimple case, there is not any isotropic root, and this theorem leads to the classification, given first by Cartan [14].

4. The isotropic roots :

As pointed out first by Kac [7] and Moody [8], the infinite dimensional structure lies in the isotropic roots ; these roots possess some particular properties, which we will point out in this subsection.

Then, we will be in position to find the complete structure of the root system of a quasisimple Lie algebra, and hence to give a general classification.

We begin with this first important lemma :

Lemma 4 :

Let α be an isotropic root of **g**. Then, for any other root β of **g**,then $<\alpha,\beta> = 0$.

The proof of this lemma is based on the following.

Proposition 5 :

Under the same assumption, if $<\alpha,\beta> \neq 0$ for a non-isotropic root β of **g**, then $\beta + n\,\alpha \in \mathbf{R}$, for infinitely many consecutive integer n.

<u>Proof :</u> Assume that there is an integer p, such that $\beta' = \beta + p\alpha$ is a root of **g**, and $\beta' - \alpha$ is not a root of **g**. Then, let $x_{\beta'}$ be a non-zero element of $\mathbf{g}_{\beta'}$.

$$[x_\alpha, x_{\beta'}] \in \mathbf{g}_{\beta'+\alpha} \text{ , for any } x_\alpha \in \mathbf{g}_{\beta'}$$

Set. $x_0 = x_{\beta'}$, and $x_n = [\mathrm{ad}(x_\alpha)]^n . x_0$

Then $[x_{-\alpha}, x_1] = [x_{-\alpha}, [x_\alpha, x_0]] = - <x_\alpha, x_{-\alpha}> <\alpha,\beta'> x_0$

Obviously $<\alpha,\beta'> = <\alpha,\beta>$, and $x_{-\alpha}$ can be chosen in $\mathbf{g}_{-\alpha}$ in such a way that $<x_\alpha, x_{-\alpha}> = 1$.

Then, we have : $[x_{-\alpha}, x_1] = - <\alpha, \beta> x_1$

Now, using the induction procedure, let us assume that

$$\mathrm{ad}(x_{-\alpha})\, x_{n-1} = -(n-1) <\alpha,\beta> x_{n-2}$$

Then : $[x_{-\alpha}, x_n] = [x_{-\alpha}, [x_\alpha, x_{n-1}]]$

$$= -[h_\alpha, x_{n-1}] - (n-1)\langle\alpha,\beta\rangle\, x_{n-1}$$

$$= -\langle\beta+(n-1)\alpha,\alpha\rangle\, x_{n-1} - (n-1)\langle\alpha,\beta\rangle\, x_{n-1}$$

$$= -n\,\langle\alpha,\beta\rangle\, x_{n-1}$$

and the proposition follows.

We are now in position to prove the lemma :

let us assume $\langle\alpha,\beta\rangle \neq 0$, and let $\gamma_n = w_{\beta'+n\alpha}.\alpha$

We then have:

$$\gamma_n = \alpha - [2\langle\alpha,\beta\rangle/(\langle\beta,\beta\rangle - 2n\langle\alpha,\beta\rangle)]\,(\beta' + n\alpha) \tag{2}$$

Using the last proposition, let us assume for example that the α-ladder of roots $\{\beta'+n\alpha\}$ does not have an upper bound ; then :

$$\alpha = \lim_{n\to\infty} \gamma_n \tag{3}$$

is a limit point in $\mathbf{R}$, and there is a contradiction with (Q.S.L.A.2), the root system being assumed to be discrete.

Then, $\langle\alpha,\beta\rangle = 0$, and the lemma is proved.

Let us denote now by $\mathbf{h'}_{\mathbb{R}}$ the real linear span of the roots, and by $\mathbf{h}_{\mathbb{R}}$ its dual space.

For any $\alpha\in\mathbf{h'}_{\mathbb{R}}$, we define $l_\alpha \in \mathbf{h}_{\mathbb{R}}$, by :

$$l_\alpha(\beta) = \langle\alpha,\beta\rangle \tag{4}$$

This then induces on $\mathbf{h}_{\mathbb{R}}$ a symmetric bilinear form : for any $\alpha, \beta \in \mathbf{h'}_{\mathbb{R}}$:

$$\langle l_\alpha, l_\beta\rangle = \langle\alpha,\beta\rangle \tag{5}$$

set

$$\mathbf{h}_* = l(\mathbf{h'}_{\mathbb{R}})$$

$$\mathbf{R'}_l = l(\mathbf{R}) \tag{6}$$

We can then state the following:

Proposition 6 :

$\langle\, ,\rangle$ is non-degenerate on $\mathbf{h}_* \times \mathbf{h}_*$.

<u>Proof :</u> let l_1, l_2 be arbitrary elements on $\mathbf{h}_*$; then, there exists α_1, $\alpha_2 \in \mathbf{h'}_{\mathbb{R}}$ such $l(\alpha_1) = l_1$ and $l(\alpha_2) = l_2$; assuming that $<l_1, l_2> = 0$ for any l_2 in $\mathbf{h}_*$ implies that $<\alpha_1, \alpha_2> = 0$, or $l_1(\alpha_2) = 0$ for any $\alpha_2 \in \mathbf{h'}_{\mathbb{R}}$, then $l_1 = 0$, and the proposition follows.

Clearly, the mapping $l.\ \mathbf{h'}_{\mathbb{R}} \to \mathbf{h}_*$ sends the isotropric part of the roots to zero. In the semisimple case, there is no isotropic root ; moreover, the Killing form is positive definite on $\mathbf{h'}_{\mathbb{R}}$, and this allows the identification of the Cartan subalgebra with its dual.

In the sequel, we shall be interested in the cases where the Killing form, when restricted to $\mathbf{h'}_{\mathbb{R}}$, is positive semidefinite ; but let us first define :

Definition 3 :

Let $\mathbf{g}$ be a quasisimple Lie algebra, and $<\,,\,>$ be its Killing form :

_ If $<\,,\,>$ is positive definite on $\mathbf{h'}_{\mathbb{R}} \times \mathbf{h'}_{\mathbb{R}}$, $\mathbf{g}$ is said to be semisimple

_ if $<\,,\,>$ is positive semidefinite on $\mathbf{h'}_{\mathbb{R}} \times \mathbf{h'}_{\mathbb{R}}$, $\mathbf{g}$ is called elliptic

_ in the other cases, $\mathbf{g}$ is called indefinite[(1)].

In the following, we will always assume that $\mathbf{g}$ is an elliptic quasisimple Lie algebra. Then, we can state the important theorem.

Theorem 7 :

$\mathbf{R'}_1$ is a finite root system, called the gradient coroot system, in the following sense :

i) If the Weyl reflexions in $\mathbf{R'}_1$ are defined by : $\forall \alpha, \beta \in \mathbf{R} : w_{l_\alpha} . l_\beta = l_{w_\alpha . \beta}$

then $\mathbf{R'}_1$ is Weyl-invariant.

ii) For any l_α, $l_\beta \in \mathbf{R'}_1$, $l_\alpha \neq 0$, $2<l_\alpha . l_\beta>/<l_\alpha . l_\alpha>$ is an integral number,

and $\quad w_{l_\alpha} . l_\beta = l_\beta - [2 <l_\alpha, l_\beta>/<l_\alpha, l_\alpha>]\, l_\alpha \qquad (7)$

iii) $\mathbf{R'}_1$ generates $\mathbf{h}_\alpha$, and is finite.

<u>Remark :</u> $\mathbf{R'}_1$ is not necessarily reduced [13]

(1): In a recent private discussion, V.G. Kac conjectured that actually the class of quasisimple Lie algebras of indefinite type could be void.

<u>Proof</u> :

i) is obvious

ii) $2<l_\alpha,l_\beta>/<l_\alpha,l_\alpha> = 2<\alpha,\beta>/<\alpha,\alpha>$ is an integral number.

For any $\gamma \in \mathbf{h'}_{\mathbb{R}}$:

$$w_{l_\alpha}.l_\beta(\gamma) = <\beta-(2<\alpha,\beta>/<\alpha,\alpha>)\alpha,\ \gamma>$$
$$= [l_\beta-(2<\alpha,\beta>/<\alpha,\alpha>l_\alpha)](\gamma)$$

and the result follows.

iii) Clearly, $\mathbf{R'}_1$ generates $\mathbf{h}_*$. Moreover, since the Killing form, restricted to $\mathbf{h}_*$ is positive definite, as a discrete subset of a compact set, $\mathbf{R'}_1$ is finite.

This concludes the proof of the theorem.

$\mathbf{R'}_1$ is then a finite root system, and will allow the description of $\mathbf{R}$.

Let $J : \mathbf{h}_* \to J(\mathbf{h}_*) = \mathbf{h'}_* \subset \mathbf{h'}_{\mathbb{R}}$ be a vector space isomorphism ; we then define the gradient root system, by

$$\mathbf{R}_1 = J(\mathbf{R'}_1) \tag{8}$$

If $\mathbf{R'}_1$ is a reduced root system, J can be chosen such that $\mathbf{R}_1 \subset \mathbf{R}$; then, we set

$$\alpha_j = J(l_j) \tag{9}$$

where the l_j are the simple roots of $\mathbf{R'}_1$. On the other hand, if $\mathbf{R'}_1$ is non-reduced, such a choice is also valid, but one needs an additional specification :

let $l_1, \dots l_n$ be the simple roots of $\mathbf{R'}_1$, and let l_n be the unique one such that $2\, l_n \in \mathbf{R'}_1$.

We set : $\alpha_j = J(l_j) \in \mathbf{R}$; the α_j are the simple roots of $\mathbf{R}_1$; since $\mathbf{R}$ is reduced, $2\alpha_n$ is an element of $\mathbf{R}_1$, but does not belong to $\mathbf{R}$. Consequently, there exists $\eta \in \mathbf{h'}_{\mathbb{R}}$ such that $<\eta,\eta>=0$, and that η is the smallest element in $\mathbf{h'}_{\mathbb{R}}$ fulfilling

$$2\,\alpha_n + \eta \in \mathbf{R} \tag{10}$$

This ends the determination of J.

Let $\mathbf{k}_{\mathbb{R}} = \mathrm{Ker}\,(l)$; we then have :

$$\mathbf{h'}_{\mathbb{R}} = \mathbf{h'}_* \oplus \mathbf{k}_{\mathbb{R}} \tag{11}$$

Let us denote :

$$\dim(\mathbf{h'}_*) = n$$

$$\dim(\mathbf{k}_{\mathbb{R}}) = \nu \qquad (12)$$

then : $\dim(\mathbf{h'}_{\mathbb{R}}) = n + \nu$

Every $\alpha \in \mathbf{h'}_{\mathbb{R}}$ can be decomposed as :

$$\alpha = (\alpha_0, \alpha_1) \qquad (13)$$

with $\alpha_1 \in \mathbf{h'}_*$, and $\alpha_0 \in \mathbf{k}_{\mathbb{R}}$.

Clearly, an element of $\mathbf{h'}_{\mathbb{R}}$ is isotropic if and only if it takes the form :

$$\alpha = (0, \alpha_0).$$

<u>A few examples</u> :

_ If $\nu=0$. $\mathbf{R}$ and $\mathbf{R}_1$ are identical ; this is the semisimple case, and it is completely solved.

_ If $\nu = 1$: this is the case of the affine Kac-Moody Lie algebras ; they can be all constructed from a generalized Cartan matrix.

_ If $\nu \geqslant 2$: the corresponding quasisimple Lie algebras are new Lie algebras ; they are not Kac-Moody Lie algebras, and they do not possess a generalized Cartan matrix, in the sense of Kac and Moody [7], [8].

Now, using **Theorem 3**, we are in position to prove the following useful proposition :

Proposition 8 :

Let α be a non-isotropic root of $\mathbf{g}$, and β an isotropic element in $\mathbf{h'}_{\mathbb{R}}$; then, if $\alpha+\beta$ is a root, so are β, $\alpha-\beta$ and $\beta-\alpha$.

<u>Proof :</u> assuming that $\alpha, \alpha+\beta \in \mathbf{R}$, then $\alpha+\beta+n\alpha$ is a root if and only if $-n_- \leqslant n \leqslant n_+$, with $n_- - n_+ = 2\langle\alpha+\beta,\alpha\rangle / \langle\alpha,\alpha\rangle = 2$.

We can deduce that $n_- \geqslant 2$, and the proposition follows.

5. Classification of the elliptic quasisimple root systems :

First, we define the irreducibility.

Definition 4 :

A root system is said to be irreducible if :

(I.R.1) : $\mathbf{R}_1$ is an irreducible if :

(I.R.2) : For any isotropic root δ, there exists an α in $\mathbf{R}_1$ such that $\alpha+\delta$ is a root ; such a root δ is called an unisolated isotropic root.

Remark : It is easy to check that any elliptic quasisimple root system can be decomposed as a disjoint union of irreducible elliptic root systems.

In the sequel, the root systems will always be assumed to be irreducible.

Let $\alpha = (\alpha_1,\alpha_0)$ be a root of $\mathbf{g}$, and $(\alpha_1,\alpha_0+\gamma_0)$ be another root of $\mathbf{g}$; let ξ_α be the isotropic element of $\mathbf{h'}_{\mathbb{R}}$ defined as follows: ξ_α is the smallest element in the straight line $\mathbb{R}\gamma_0$ such that $(\alpha_1,\alpha_0+\xi_\alpha)$ is a root. Similarly, in the case of a non-reduced $\mathbf{R}_1$, let β_1 be a short root in $\mathbf{R}_1$, and let $\beta=(\beta_1,\beta_0)$ be a root of $\mathbf{g}$; we define ξ'_β to be the smallest point in $\mathbb{R}\gamma_0$ such that $(2\beta_1, 2\beta_0 + \xi'_\beta) \in \mathbf{R}$ (In the following, we shall see that the existence of ξ'_β is easily checked). Then, we can prove :

Lemma 9 :

One has the three following assertions:

i) $(\alpha_1,\alpha_0+n\xi_\alpha)$ is a root for any $n \in \mathbb{Z}$

ii) If $\mathbf{R}_1$ is non-reduced : $(2\beta_1, 2\beta_0 + r\xi'_\beta)$ is a root if and only if r is an odd integer.

iii) $\xi'_\beta = \xi_\beta$ for any short root β of $\mathbf{R}_1$

Proof : i) Let $\delta = (\alpha_1, \alpha_2+\xi_\alpha)$.

Then $w_\delta.w_\alpha.(\alpha_1,\alpha_0+r\xi_\alpha) = (\alpha_1, \alpha_0+ (2+r)\xi_\alpha)$ and

$$(w_\delta.w_\alpha)^m.(\alpha_1,\alpha_0+ r\xi_\alpha) = (\alpha_1,\alpha_0+ (2m+r)\xi_\alpha) \tag{14}$$

Considering (14) with $r = 0,1$ shows that $(\alpha_0,\alpha_1+ k\xi_\alpha)$ is a root for any integer k. Conversely, m can be chosen in (14) in such a way that

$$\alpha_0-\xi_\alpha < \alpha_0+ (2m+r)\xi_\alpha \leqslant \alpha_0+\xi_\alpha ,$$

which contradicts the minimality of ξ_α, if $2m+r$ is different from 0 or 1 ;

i) is then checked.

ii) Let $(2\beta_1, 2\beta_o + \xi'_\beta)$ be the root of **g** defined in the lemma :

$$w_\beta.(2\beta_1, 2\beta_o+\xi'_\beta) = (-2\beta_1, -2\beta_o+\xi'_\beta) \in \mathbf{R} \tag{15}$$

Then $(2\beta_1, 2\beta_o - \xi'_\beta) \in \mathbf{R}$ and applying i), ii) follows.

iii) $w_{2\beta+\xi'_\beta}.\beta = -(\beta_1, \beta_o + \xi'_\beta) \in \mathbf{R}$

Then, from i), $\xi'_\beta = n\xi_\beta$, with $n \in \mathbb{N}\backslash\{o\}$; **R** being reduced, n is obviously odd ; assuming $n \geqslant 3$, there exists $m \in \mathbb{Z}$ such that $n/4 < m < n/2$

$$w_{\beta+m\xi_\beta}.(2\beta + \xi'_\beta) = -2\beta + (4m-n)\,\xi_\beta \in \mathbf{R} \tag{16}$$

with $0 < (4m-n) < n$, which leads to a contradiction with the definition of ξ'_β ; hence, $n = 1$, and iii) and the lemma follow . (this proof is identical to Mc Donald's proof [2]).

Since we are interested in irreducible quasisimple root systems, the Dynkin diagram of the gradient root system has to be a connected graph.

Let us consider an arbitrary subdiagram, with only two vertices :

$$\underset{\alpha_1}{\circ} = \overset{k}{<} = \underset{\alpha_2}{\circ}$$

($k = 1,2$ or 3), and let us assume that there exists an isotropic root ξ_{α_1} such that $(\alpha_1, \xi_{\alpha_1}) \in \mathbf{R}$, and ξ_{α_1} is the smallest root on the straight line $\mathbf{R}^+\xi_{\alpha_1}$ having this property. Then

$$w_{\alpha_1}.w_{\alpha_1+\xi_{\alpha 1}}.\alpha_2 = \alpha_2 + k\,\xi_{\alpha_2} \in \mathbf{R} \tag{17}$$

and one has:

Proposition 10 :

There exist isotropic roots $\delta = \lambda\xi_{\alpha_1} (\lambda \in \mathbb{Z})$ such that $\alpha_2+\delta$ is a root of **g**.

Let ξ_{α_2} be the smallest of these roots ; then we have

$$k\,\xi_{\alpha_1} \in \mathbb{N}\,\xi_{\alpha_2} \tag{18}$$

Similarly, we deduce from

$$w_{\alpha_2}.w_{\alpha_2+\xi_{\alpha 2}}.\alpha_1 = \alpha_1 + \xi_{\alpha_2} \in \mathbf{R} \tag{19}$$

that

$$\xi_{\alpha_2} \in \mathbb{N}\, \xi_{\alpha_1} \tag{20}$$

Then one can state:

Proposition 11 :

Assuming that ξ_{α_1} is known, we have only two possibilities for ξ_{α_2} :

i) $\xi_{\alpha_2} = \xi_{\alpha_1}$: non-twisted case

ii) $\xi_{\alpha_2} = k\, \xi_{\alpha_1}$: twisted case

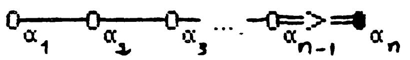
Remark : clearly, when the Dynkin diagram of $\mathbf{R}_1$ has only simple links (i.e. all roots have the same length), there is no twisted case.

Corollary :

Let ξ be an isotropic root of $\mathbf{g}$, and let α be a root of $\mathbf{g}$ of minimal length ; then $\alpha + \xi \in \mathbf{R}$

Remark : In the ν-dimensional isotropic subspace of $\mathbf{h'}_{\mathbb{R}}$, we are not able to distinguish the different directions ; we can only precise the number of twists.

The most complicated case happens when the gradient root system is non-reduced ; hence, $\mathbf{R}_1$ is of the form $\mathbf{BC}_n$, and has the Dynkin diagram :



where $2\alpha_n \in \mathbf{R}_1$ too. In this case, from our choice of J, $\alpha_1, \dots \alpha_n \in \mathbf{R}$, and $2\alpha_n + \eta \in \mathbf{R}$; let us decompose η with respect to a basis of $\mathbf{k}_{\mathbb{R}}$:

$$\eta = \sum_1^{\nu} \eta^i \tag{21}$$

Obviously :

_ Since $\mathbf{R}$ is reduced, there exists at least one i in $\{1, \dots \nu\}$ such that $\eta^i \neq 0$

_ Since $2\alpha_n + \eta \in \mathbf{R}$ and $2\alpha_n - \eta \in \mathbf{R}$, then we have $\eta^i = \xi^i_{2\alpha_n+\eta}/2$ or $\eta^i = 0$.

To simplify the notations, let us denote $\xi^i_{2\alpha_n} = \xi^i$.

The same proof than in prop. 10-11 leads to :

$$\xi^i_{\alpha_1} = \xi^i_{\alpha_2} = \dots = \xi^i_{\alpha_{n-1}} = \xi^i \text{ or } 2\xi^i$$
$$\text{If } \xi^i_{\alpha_1} = \xi^i \quad \text{then} \quad \xi^i_{2\alpha_n+\eta} = \xi^i \text{ or } 2\xi^i \tag{22}$$
$$\text{If } \xi^i_{\alpha_1} = 2\xi^i \quad \text{then} \quad \xi^i_{2\alpha_n+\eta} = 2\xi^i \text{ or } 4\xi^i$$

Hence, we will have to consider the following different situations (α_L and α_S denote respectively a long and a short root) :

– $\xi^i_{2\alpha_n+\eta} = \xi^i_{2\alpha_L} = \xi^i$; then $\eta^i = 0$

– $\xi^i_{2\alpha_n+\eta} = 2\,\xi^i$; the roots are then :

$$\alpha_S + n\,\xi^i \qquad \forall n \in \mathbb{Z}$$
$$\alpha_L + n\,\xi^i_L \qquad \forall n \in \mathbb{Z}$$
$$2\alpha_S + \eta + 2n\,\xi^i \qquad \forall n \in \mathbb{Z}$$

$2\langle 2\alpha_S+\eta+2n\xi^i, \alpha_L\rangle / \langle \alpha_L,\alpha_L\rangle = -2$: then $2\alpha_S + \alpha_L + \eta + 2n\xi^i \in \mathbf{R} \quad \forall n\in\mathbb{Z}$

$2\langle 2\alpha_S+\alpha_L+\eta+2n\xi^i, \alpha_S\rangle / \langle \alpha_S,\alpha_S\rangle = 2$: then $\alpha_L + \eta + 2n\xi^i \in \mathbf{R} \quad \forall n\in\mathbb{Z}$

If $\xi^i_L = 2\xi^i$: then $\eta^i = 0$

If $\xi^i_L = \xi^i$: then $\eta^i = 0$ or ξ^i

– $\xi^i_{2\alpha_n+\eta} = 4\,\xi^i$: then $\eta^i = 0$ or $2\xi^i$

but these two solutions being equivalent, we choose $\eta^i = 0$.

All these results are collected in the following classification theorem :

Theorem 12 :

Let **R** be an irreducible elliptic quasisimple root system, and let $\mathbf{R}_1$ be its gradient root system

i) If $\mathbf{R}_1$ is reduced, then **R** can be completely characterized by the expression

$$\mathbf{R} \simeq (\,\mathbf{R}_1\,;\nu\,,\tau\,) \tag{23}$$

with $\nu = \dim \mathbf{k}_{\mathbb{R}}$ and τ = number of twists

An arbitrary root takes then the form :

$$\alpha_S + \textstyle\sum_1^\nu n_i\,\xi^i$$
$$\alpha_L + \textstyle\sum_1^\tau 2n_i\,\xi^i + \sum_{\tau+1}^\nu m_i\,\xi^i$$
$$\textstyle\sum_1^\nu n_i\,\xi^i \qquad\qquad \forall m_i, n_i\in\mathbb{Z}$$

ii) If $\mathbf{R}_1 = \mathbf{BC}_n$, $\mathbf{R}$ is completely characterized by :

$$\mathbf{R} \simeq (\mathbf{BC}_n ; \nu ; \tau_1, \tau_2, \tau_3, \tau_4) \qquad 1 \leqslant \tau_1 \leqslant \nu$$

refering to table I ; the roots are :

$$\alpha_S + \sum_1^{\nu} n_i \, \xi^i$$

$$\alpha_L + \sum_1^{\tau_1+\tau_2} n_i \, \xi^i + \sum_{\tau_1+\tau_2+1}^{\nu} 2m_i \, \xi^i$$

$$2\alpha_S + \sum_1^{\tau_1} (2n_i+1) \, \xi^i + \sum_{\tau_1+1}^{\tau_1+\tau_3} m_i \, \xi^i + \sum_{\tau_1+\tau_3+1}^{\tau_1+\tau_2+\tau_4} 2p_i \, \xi^i + \sum_{\tau_1+\tau_2+\tau_4+1}^{\nu} 4q_i \, \xi^i$$

$$\sum_1^{\nu} n_i \, \xi^i \qquad \forall \, n_i, m_i, p_i, q_i \in \mathbb{Z}$$

Table I :

<table>
<tr><td>$\eta^i = \xi^i$
$i = 1, \dots \tau_1$</td><td>$\xi^i_{\alpha_L} = \xi^i$</td><td>$\xi^i_{2\alpha_S+\eta} = 2\,\xi^i$</td></tr>
<tr><td rowspan="4">$\eta^i = 0$
$i = \tau_1 + 1, \dots \nu$</td><td rowspan="2">$\xi^i_{\alpha_L} = \xi^i$
$i = \tau_1 + 1, \dots \tau_1 + \tau_2$</td><td>$\xi^i_{2\alpha_S+\eta} = \xi^i$
$i = \tau_1 + 1, \dots \tau_3 + \tau_1$</td></tr>
<tr><td>$\xi^i_{2\alpha_S+\eta} = 2\xi^i$
$i = \tau_1 + \tau_3 + 1, \dots \tau_1 + \tau_2$</td></tr>
<tr><td rowspan="2">$\xi^i_{\alpha_L} = 2\xi^i$
$i = \tau_1 + \tau_2 + 1, \dots \nu$</td><td>$\xi^i_{2\alpha_S+\eta} = 2\xi^i$
$i = \tau_1 + \tau_2 + 1, \dots \tau_1 + \tau_2 + \tau_3$</td></tr>
<tr><td>$\xi^i_{2\alpha_S+\eta} = 4\xi^i$
$i = \tau_1 + \tau_2 + \tau_3 + 1, \dots \nu$</td></tr>
</table>

<u>Conclusive remarks</u> : In this section, we have classified all possible irreducible elliptic quasisimple root systems ; however, this classification does not extend to a complete classification of the quasisimple Lie algebras. We have not proved that the root system completely determines a Lie algebra; this is probably related to the fact that in the case $\nu \geqslant 2$ (the case $\nu = 0$ or 1 are well known), the Weyl group is no longer a Coxeter group (see [13]). Nevertheless, it is possible to compute the root-multiplicities, and to prove that an irreducible elliptic quasisimple Lie algebra of type ν possesses a ν-dimensional center ; it is possible to investigate the possible Killing forms; for such an analysis, we refer to [15].

III. <u>CONSTRUCTION :</u>

After having classified them, we are able to prove that some of the new infinite-dimensional Lie algebras exist. We can build irreducible elliptic QSLA's for arbitrary ν and $\tau = 0$, 1 or 2, using techniques familiar from the affine case, i.e. realisations as current algebras (see [1] for instance).

<u>1. The non-twisted case :</u>

Let $\mathbf{g}_0$ be a simple complex Lie algebra, with Cartan subalgebra $\mathbf{h}_0$, and root system $\mathbf{R}_0$; let $\{\alpha_1, \ldots \alpha_n\}$ be the simple roots of $\mathbf{R}_0$; let T^ν be the ν-dimensional torus, provided with the Lebesgue measure dt, such that :

$$\int_{T^\nu} dt = 1 \tag{24}$$

Let $\mathbf{g}_1 = P(T^\nu, \mathbf{g}_0)$ be the Lie algebra of functions with finite Fourier series, with the pointwise Lie structure : $\forall\, x,y \in \mathbf{g}_1$

$$[x,y]_1(t) = [x(t),y(t)]_0 \qquad \forall\, t \in T^\nu$$

$$\langle x,y\rangle_1 = \int_{T^\nu} \langle x(t),y(t)\rangle_0 dt \tag{25}$$

(2) : Actually, V.G. Kac suggested that for $\nu > 2$, there could be many Lie algebras corresponding to a given root system ; this is an interesting still open problem.

In order to get a finite-dimensional Cartan subalgebra, we define the derivations $D(d)$ of $\mathbf{g}_1$ ($d \in \mathbb{C}^\nu$) by : $\forall\, x \in \mathbf{g}_1$

$$D(d).x = (d.\nabla).x \tag{26}$$

and $\mathbf{g}_2$ to be the semidirect product of $\mathbf{g}_1$ by $\mathbb{C}^\nu$;

with commutation relations : $\forall\, x,x' \in \mathbf{g}_1$, $\forall\, d,d' \in \mathbb{C}^\nu$

$$[(x,d),(x',d')]_2 = ([x,x']_1 + D(d).x' - D(d').x\ ,\ 0) \tag{27}$$

A Cartan subalgebra of $\mathbf{g}_2$ can be written in the form :

$$\mathbf{h}_2 = \mathbf{h}_0 \oplus \textstyle\sum_1^\nu \mathbb{C}d_i \tag{28}$$

where the d_i's form a basis of $\mathbb{Z}^\nu$.

Finally, to get a Killing form, we define on $\mathbf{g}_1$ the following family of $\mathbb{C}$-valued bilinear forms : $\forall\, d \in \mathbb{C}^\nu$

$$\psi_d(x,y) = \langle x, D(d).y\rangle_1 \qquad \forall\, x,y \in \mathbf{g}_1 \tag{29}$$

Proposition 13 :

For any d in $\mathbb{C}^\nu$, ψ_d is a Chevalley 2-cocycle :

i) $\psi_d(x,y) = \psi_d(y,x) \quad \forall\, x,y \in \mathbf{g}_1$

ii) $\psi_d([x,y],z) + \psi_d([y,z)],x) + \psi_d([z,x],y) = 0 \quad \forall\, x,y,z \in \mathbf{g}_1$.

The proof follows easily from the invariance of $<,>_0$. Consequently, it is possible to make a central extension with respect to the ψ_d ; let then $\mathbf{C}$ be a ν-dimensional center, spanned by complex linear combinations of the vectors $c_1, \dots c_\nu$. Then, we define a Lie algebra:

$$\mathbf{g} = P(T^\nu,\mathbf{g}_0) \oplus \textstyle\sum_1^\nu \mathbb{C}d_i \oplus \sum_1^\nu \mathbb{C}c_i \tag{30}$$

with Lie bracket: $\forall\, \{x,d,c\},\{x',d',c'\} \in \mathbf{g}$

$$[\{x,d,c\},\{x',d',c'\}] = \{[x,x']_1 + D(d).x' - D(d').x\ ,\ 0\ ,\ \textstyle\sum_1^\nu \psi_{d_i}(x,x').c_i\} \tag{31}$$

It remains to define the Killing form ; we set : $\forall\, \{x,d,c\},\ \{x',d',c'\} \in \mathbf{g}$

$$\langle\{x,d,c\},\{x',d',c'\}\rangle = \langle x,x'\rangle_1 - d.c' - d'.c \tag{32}$$

where . is the usual scalar product in $\mathbb{C}^\nu$.

Proposition 14 :

$<,>$ is a non degenerate symmetric invariant bilinear form on $\mathbf{g}$.

<u>Proof.</u> non degeneracy and symmetry are obvious.

Let $\{x,d,c\}, \{x',d',c'\}, \{x'',d'',c''\} \in \mathbf{g}$

$<[\{x,d,c\}, \{x',d',c'\}], \{x'', d'', c''\}>$

$$= <[x,x']_1 + D(d).x' - D(d').x,\ 0,\ \sum_1^\nu \psi_{d_i}(x,x').c_i\}, \{x'',d'',c''\}>$$

$$= <[x,x']_1, x''>_1 + \psi_d(x'',x') - \psi_{d'}(x'',x) - \psi_{d''}(x,x')$$

$<\{x,d,c\}, [\{x',d',c'\}, \{x'',d'',c''\}]>$

$$= <\{x,d,c\}, \{[x',x'']_1 + D(d').x'' - D(d'').x',\ 0,\ \sum_1^\nu \psi_{d_i}(x',x'').c_i\}>$$

$$= <x, [x', x'']_1>_1 + \psi_{d'}(x,x'') - \psi_{d''}(x,x') - \psi_d(x',x'')$$

and the proposition follows from the invariance of $<,>_1$.

$<,>$ is then a relevant Killing form for $\mathbf{g}$.

The Cartan subalgebra takes the form :

$$\mathbf{h} = \mathbf{h}_o \oplus \sum_1^\nu \mathbb{C}d_i \oplus \sum_1^\nu \mathbb{C}c_i \qquad (33)$$

and we easily see that $\mathbf{g}$ is a Q.S.L.A..

We now have to study its root system, i.e. the spectrum of ad($\mathbf{h}$)

- $\forall c \in \mathbf{C}$, $ad(c) = 0$
- Let $d = \sum_1^\nu \lambda_i d_i$; we have $ad(d) = D(d)$

and we have to solve :

$$D(d).X = m(d)\, X \quad \text{with } X \in \mathbf{g} \qquad (34)$$

The solutions are of the form :

$$X_m : t \in T^\nu \to X_m(t) = [\prod_1^\nu \exp\{im_k t_k\}].E \quad \text{with } E \in \mathbf{g}_o$$

$$m(d) = i \sum_1^\nu \lambda_k m_k \qquad (35)$$

where $m = (m_1, \dots m_\nu) \in \mathbb{Z}^\nu$

- for any $h \in \mathbf{h}_o$:

$$ad(h).\, X_{m,\alpha} = \alpha(h)\, X_{m,\alpha}$$

where $X_{m,\alpha} = [\prod_1^\nu \exp\{im_k t_k\}].E_\alpha$; $E_\alpha \in (\mathbf{g}_o)_\alpha$

The spectrum of ad(h) is then given by : $\forall \; \{h,d,c\} \in \mathbf{h}$

$$\mathrm{ad}(\{h,d,c\}).X_{m,\alpha} = [m(d) + \alpha(h)]\, X_{m,\alpha} \tag{36}$$

The Killing form is canonically carried onto $\mathbf{h}$, by defining, for any $\gamma \in \mathbf{h'}$, the canonical identification.

$$\langle e(\gamma), h\rangle = \gamma(h) \tag{37}$$

Then, if $\alpha + m$ is any root of $\mathbf{g}$:

$$e(\alpha + m) = \{\, h_\alpha \,,\, 0 \,,\, i\textstyle\sum_1^\nu m_k c_k \,\} \tag{38}$$

and m is easily identified as being the isotropic part of the root $\alpha + m$.

Moreover, it is easy to check that $\langle\, ,\rangle$ is positive semidefinite of corank ν on $\mathbf{h'}_{\mathbb{R}} \times \mathbf{h'}_{\mathbb{R}}$, and it follows :

Theorem 15 :

$\mathbf{g}$ is an elliptic quasisimple Lie algebra, associated with the root system :

$$\mathbf{R} = (\mathbf{R}_0 \,;\, \nu,\, 0\,)$$

Any non-twisted elliptic Q.S.L.A. can then be realized in this way.

2. The case $\tau = 1$:

The construction given there is fairly standard, and can be found for instance in Kac [1].

We consider a simple Lie algebra $\mathbf{g}_0$, whose Dynkin diagram admits a non trivial automorphism σ, of order k ; let now ρ be an automorphism of order k of the torus T^ν, for example :

$$t_1 \to t_1 + 2\pi/k$$

$$t_i \to t_i \qquad \forall\, i \neq 1$$

We define the automorphism $\tilde{\sigma}$ on $\mathbf{g}$ by :

$$\begin{aligned} &\tilde{\sigma}(X) = \sigma(X \circ \rho) && \forall\, X \in \mathbf{g}_1 \\ &\tilde{\sigma}(c_i) = c_i && \forall\, i = 1,\dots\nu \\ &\tilde{\sigma}(d_i) = d_i && \forall\, i = 1,\dots\nu \end{aligned} \tag{39}$$

Let $\tilde{\mathbf{g}}$ be the fixed point set of $\tilde{\sigma}$; we have :

$$\tilde{\mathbf{g}} = \sum_1^k P_i(T^\nu, \mathbf{g}_0^{(-i)}) \oplus \sum_1^\nu \mathbb{C}d_i \oplus \sum_1^\nu \mathbb{C}c_i \tag{40}$$

where $\mathbf{g}_0^{(l)} = \{ x \in \mathbf{g}_0 \,/\, \rho.x = \exp(i2\pi l/k)\, x \}$

and $\quad P_1(T^\nu, \mathbf{g}_0^{(l)}) = \{ x \in P(T^\nu, \mathbf{g}_0^{(-l)})$ s.t. $[\sigma(x)] = x(t) \;\; \forall t \in T^\nu \}$

There are five possible automorphisms for the $\mathbf{g}_0$ given by the five possible QSLA's $(\mathbf{R}_1 ; \nu, 1)$ or $(\mathbf{BC}_n ; \nu ; 1, 1 ; 0, 0)$; in each case, it is easily seen that $\mathbf{g}_0^{(o)}$ is a simple Lie algbra :

– $\mathbf{g}_0 = \mathbf{A}_{2n} : \sigma^2 = 1 \quad \mathbf{g}_0^{(o)} = \mathbf{B}_n \qquad\qquad (\mathbf{A}_2^{(o)} = \mathbf{A}_1)$

$\qquad R(\tilde{\mathbf{g}}) \simeq (\mathbf{BC}_n ; \nu ; 1, 1, 0,0)$

– $\mathbf{g}_0 = \mathbf{A}_{2n-1} : \sigma^2 = 1 \quad \mathbf{g}_0^{(o)} = \mathbf{C}_n$

$\qquad R(\tilde{\mathbf{g}}) \simeq (\mathbf{C}_n ; \nu ; 1)$

– $\mathbf{g}_0 = \mathbf{D}_{n+1} : \sigma^2 = 1 \quad \mathbf{g}_0^{(o)} = \mathbf{B}_n$

$\qquad R(\tilde{\mathbf{g}}) \simeq (\mathbf{BC}_n ; \nu ; 1, 1, 0,0)$

– $\mathbf{g}_0 = \mathbf{D}_4 : \sigma^3 = 1 \quad \mathbf{g}_0^{(o)} = \mathbf{G}_2$

$\qquad R(\tilde{\mathbf{g}}) \simeq (\mathbf{G}_2 ; \nu ; 1,)$

– $\mathbf{g}_0 = \mathbf{E}_6 : \sigma^2 = 1 \quad \mathbf{g}_0^{(o)} = \mathbf{F}_4$

$\qquad R(\tilde{\mathbf{g}}) \simeq (\mathbf{F}_4 ; \nu ,0)$

(The proof of this result can be found in Kac [1])

This leads to the following :

Theorem 16 :

Any irreducible elliptic quasisimple root system of twist on can be realized as a "generalized loop algebra", as constructed in section **III–2.**

3. The case $\tau = 2$ (reduced $\tilde{R}_1$) :

In this subsection, we shall see that it is possible to build a Q.S.L.A. for any irreducible elliptic quasisimple root system, very similar to the one that leads to the twist one Q.S.L.A's.

First of all, let $\mathbf{g}_0$ be an affine Kac-Moody Lie algebra ($\nu = 1$), and let us define :

$$\hat{\mathbf{g}}_0 = \mathbf{g}_0/\mathbb{C}d_1 \oplus \mathbb{C}c_1 \tag{41}$$

with the usual notations.

It is fairly easy to see that

$$\mathbf{g} = P(T^{\nu-1}, \hat{\mathbf{g}}_0) \oplus \sum_1^\nu \mathbb{C}di \oplus \sum_1^\nu \mathbb{C}c_i \tag{42}$$

provided with the usual Lie product and Killing form is an elliptic Q.S.L.A. , with root system:

$$\mathbf{R} \simeq (\mathbf{R}_1(\mathbf{g}_0)\ ;\ \nu,\ \tau(\mathbf{g}_0)) \qquad (\tau = 0 \text{ or } 1)$$

if $\mathbf{R}_1\,(\mathbf{g}_0)$ is reduced, and the corresponding one if $\mathbf{R}_1$ is non-reduced :

$$\mathbf{R} \simeq (\mathbf{BC}_n\ ;\ \nu\ ;\ 1,\ \nu-1,\ \nu-1,\ 0).$$

Now, let σ be the straight extension [12] of an automorphism of the Dynkin diagram of $\mathbf{g}_0$; it is easy to see that σ acts trivially on c_1 and d_1 ; let k be the order of σ, and let ρ be an automorphism of order k of $T^{\nu-1}$.

We define the automorphism $\tilde{\sigma}$ of $\mathbf{g}$ by :

$$\begin{aligned} &\tilde{\sigma}(X) = \sigma(X \circ \rho) && \forall\, X \in P(T^{\nu-1}, \mathbf{g}_0) \\ &\tilde{\sigma}(c_i) = c_i && \forall\, i = 1, \dots \nu-1 \\ &\tilde{\sigma}(d_i) = d_i && \forall\, i = 1, \dots \nu-1 \end{aligned} \tag{43}$$

and $\tilde{\mathbf{g}}$ is the fixed point set of $\tilde{\sigma}$; it is a Q.S.L.A., and we will look for its root system.

In the sequel, we will use the notations :

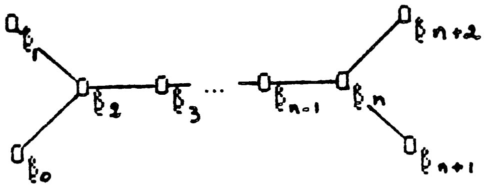
E_i, F_i, $\beta_i^\vee$ are the Chevalley generators of $\mathbf{g}_0$

β_i are the roots of $\mathbf{g}$.

e_i, f_i, $\alpha_i^\vee$ are the Chevalley generators of $\tilde{\mathbf{g}}$

α_i are the roots of $\tilde{\mathbf{g}}$.

i) $\mathbf{g}_0 = (\mathbf{D}_{n+2}\,;\,1,0)$:



$\sigma(\beta_1) = \beta_0\ ;\ \sigma(\beta_0) = \beta_1$

$\sigma(\beta_{n+1}) = \beta_{n+2}\ ;\ \sigma(\beta_{n+2}) = \beta_{n+1}\ ;\ \sigma(\beta_i) = \beta_i\quad (i=2, \dots n)$

We set :

$$\alpha_0^\vee = \beta_0^\vee + \beta_1^\vee \qquad e_0 = E_0 + E_1 \qquad f_0 = F_0 + F_1$$

$$\alpha_i^\vee = \beta_{i+1}^\vee \qquad e_i = E_{i+1} \qquad f_i = F_{i+1} \qquad (i=2, \dots n)$$

$$\alpha_n^\vee = \beta_{n+1}^\vee + \beta_{n+2}^\vee \qquad e_n = E_{n+1} + E_{n+2} \qquad f_n = F_{n+1} + F_{n+2}$$

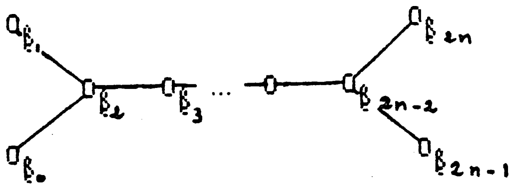
These are the Chevalley generators of

$$\mathbf{g}_0^{(o)} = (\mathbf{B}_n\,;\,1,1)$$

Now, using the analogue of (40), we are able to build the corresponding $\tilde{\mathbf{g}}$, and its root system reveals to be :

$$\mathbf{R}(\tilde{\mathbf{g}}) = (\mathbf{B}_n\,;\,\nu,2)$$

ii) $\mathbf{g}_0 = (\mathbf{D}_{2n}\,;\,1,0)$



$\sigma(\beta_i) = \beta_{2n-i} \qquad i = 0, \dots 2n$

We set :

$$\alpha_i^\vee = \beta_i^\vee + \beta_{2n-i}^\vee \qquad e_i = E_i + E_{2n-i} \qquad f_i = F_i + F_{2n-i}$$

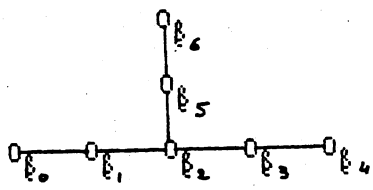
These are the Chevalley generators of

$$\mathbf{g}_0^{(o)} = (\mathbf{C}_n\,;\,1,1)$$

It is then easy to find the root system of $\tilde{\mathbf{g}}$

$$\mathbf{R}(\tilde{\mathbf{g}}) \simeq (\mathbf{C}_n\,;\,1,1)$$

iii) $\mathbf{g}_0 = (\mathbf{E}_6\,;\,1,0)$:



$\sigma(\beta_2) = \beta_2$

$\sigma(\beta_0) = \beta_6\ ;\ \sigma(\beta_6) = \beta_4\ ;\ \sigma(\beta_4) = \beta_0$

$\sigma(\beta_1) = \beta_5\ ;\ \sigma(\beta_5) = \beta_3\ ;\ \sigma(\beta_3) = \beta_1$

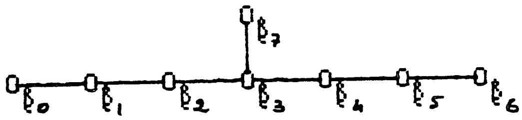
$$\alpha_0^\vee = \beta_0^\vee + \beta_4^\vee + \beta_6^\vee \qquad e_0 = E_0 + E_4 + E_6 \qquad f_0 = F_0 + F_4 + F_6$$
$$\alpha_2^\vee = \beta_1^\vee + \beta_3^\vee + \beta_5^\vee \qquad e_1 = E_1 + E_3 + E_5 \qquad f_1 = F_1 + F_3 + F_5$$
$$\alpha_2^\vee = \beta_2^\vee \qquad e_2 = E_2 \qquad f_2 = F_2$$

are the Chevalley generators of

$$\mathbf{g}_0^{(o)} = (\mathbf{G}_2\, ;\, 1,\, 1)$$

and we easily check that $\mathbf{R}(\tilde{\mathbf{g}}) \simeq (\mathbf{G}_2\, ;\, \nu,\, 2)$

iv) $\mathbf{g}_0 = (\mathbf{E}_7\, ;\, 1,\, 0)$



$\sigma(\beta_i) = \beta_{6-i} \qquad (i = 0, \ldots 6)$

$\sigma(\beta_7) = \beta_7$

Then

$$\alpha_i^\vee = \beta_i^\vee + \beta_{6-i}^\vee \qquad e_i = E_i + E_{6-i} \qquad f_i = F_i + F_{6-i}\ (i=0,\ldots 3)$$
$$\alpha_4^\vee = \beta_7^\vee \qquad e_4 = E_7 \qquad f_4 = F_7$$

are the Chevalley generators of :

$$\mathbf{g}_0^{(o)} = (\mathbf{F}_4\, ;\, 1,\, 1)$$

and we get :

$$\mathbf{R}(\tilde{\mathbf{g}}) \simeq (\mathbf{F}_4\, ;\, \nu,\, 2)$$

i) – iv) lead to the following :

Theorem 17 :

Any irreducible elliptic quasisimple root system, with reduced gradient root system, of twist two, can be realized as twisted extension of an affine Lie algebra.

We will now get a similar result in the case when $\tilde{R}_1$ is not reduced.

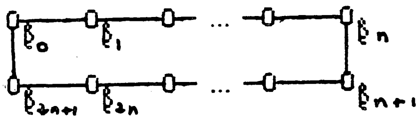
4. The case $\tilde{R}_1$ non-reduced :

i) First of all, take $\mathbf{g}_o = (\mathbf{BC}_n\ ;\ 1,\ 1,\ 0,0,0)$, and make the non-twisted extension $\mathbf{g}$ of (42) ; the root system of $\tilde{\mathbf{g}}$ is clearly :

$$R(\tilde{\mathbf{g}}) = (\mathbf{BC}_n\ ;\ \nu\ ;\ 1,\ \nu-1,\ \nu-1,\ 0)$$

The other QSLA's with non-reduced gradient root system will be built making twisted extensions of affine Lie algebras, like in the subsection 3.

ii) $\mathbf{g}_o = (\mathbf{A}_{2n+1}\ ;\ 1,\ 0)$



$\sigma(\beta_i) = \beta_{2n+1-i}$

$$\alpha_o^\vee = 2(\beta_o^\vee + \beta_{2n+1}^\vee) \qquad e_o = 2(E_o+E_{2n+1}) \qquad f_o = F_o + F_{2n+1}$$
$$\alpha_n^\vee = 2(\beta_n^\vee + \beta_{n+1}^\vee) \qquad e_n = 2(E_n+E_{n+1}) \qquad f_n = F_n + F_{n+1}$$
$$\alpha_o^\vee = 2(\beta_o^\vee + \beta_{2n+1-i}^\vee) \qquad e_i = E_i+E_{2n+1-i} \qquad f_i = F_i + F_{2n+1-i} \quad (i=2,...n-1)$$

are the Chevalley generators of :

$$\mathbf{g}_o^{(o)} = (\mathbf{B}_n\ ;\ 1,\ 1)$$

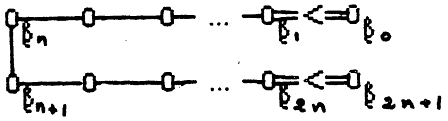
$\mathbf{g}_o^{(1)}$ is an irreducible $\mathbf{g}_o^{(o)}$-module, and the roots of $\tilde{\mathbf{g}}$ are, by a simple calculation, shown to be of the form :

$$\textstyle\sum_1^\nu\ n_i\,\xi^i \qquad\qquad \forall\ n_i \in \mathbb{Z}$$
$$\alpha_S + \textstyle\sum_1^\nu n_i\,\xi^i \qquad\qquad \forall\ n_i \in \mathbb{Z}$$
$$\alpha_L + n_1\xi^1 + 2n_2\xi^2 + \textstyle\sum_3^\nu\ n_i\,\xi^i \qquad\qquad \forall\ n_i \in \mathbb{Z}$$
$$2\alpha_S + (2n_1+1)\xi^1 + 2n_2\xi^2 + \textstyle\sum_3^\nu\ n_i\,\xi^i \qquad\qquad \forall\ n_i \in \mathbb{Z}$$

Then :

$$R(\tilde{\mathbf{g}}) \simeq (\mathbf{BC}_n\ ;\ \nu\ ;\ 1,\ \nu-2,\ \nu-2,\ 1)$$

iii) $\mathbf{g}_o = (\mathbf{C}_{2n+1}\ ;\ 1,\ 0)$



$\sigma(\beta_i) = \beta_{2n+1-i}$

$$\alpha_i^\vee = \beta_i^\vee + \beta_{2n+1-i}^\vee \qquad e_i = E_i+E_{2n+1-i} \qquad f_i=F_i+F_{2n+1-i} \quad (i=0,...n-1)$$
$$\alpha_n^\vee = 2(\beta_n^\vee + \beta_{n+1}^\vee) \qquad e_n = 2(E_n+E_{n+1}) \qquad f_n = F_n+F_{n+1}$$

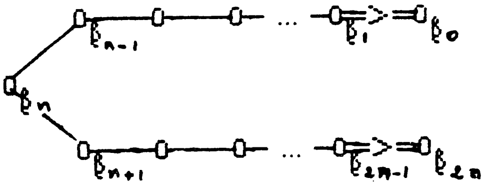
are the Chevalley generators of the affine Lie algebra

$$\mathbf{g}_o^{(o)} = (\mathbf{BC}_n\ ;\ 1\ ;\ 1,\ 0,\ 0,\ 0)$$

The usual calculations lead to :

$$R(\widetilde{g}) \simeq (BC_n \,;\, \nu \,;\, 2, \nu-2, \nu-2, 0)$$

iv) $g_o = (B_{2n} \,;\, 1, 1)$



$\sigma(\beta_i) = \beta_{2n-i} \qquad (i = 0, \ldots n)$

$$\alpha_i^\vee = \beta_i^\vee + \beta_{2n-i}^\vee \qquad e_i = E_i + E_{2n-i} \qquad f_i = F_i + F_{2n-i} \qquad (i=0,\ldots n-1)$$

$$\alpha_n^\vee = \beta_n^\vee \qquad e_n = E_n \qquad f_n = F_n$$

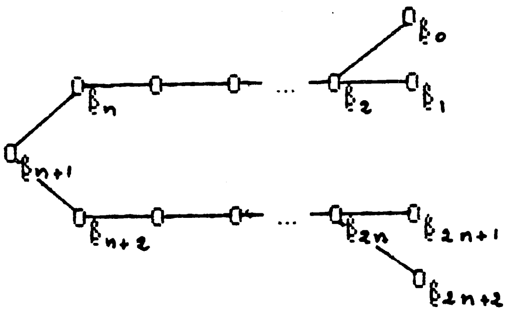
are the Chevalley generators of :

$$g_o^{(o)} = (BC_n \,;\, 1 \,;\, 1, 0, 0, 0)$$

The explicit calculation of the root system leads to :

$$R(\widetilde{g}) \simeq (BC_n \,;\, \nu \,;\, 1, \nu-1, \nu-2, 0)$$

v) $g_o = (D_{2n+2} \,;\, 1, 0)$



$\sigma(\beta_i) = \beta_{2n+2-i} \qquad (i = 3, \ldots 2n)$

$\sigma(\beta_o) = \beta_{2n+1} \,;\, \sigma(\beta_{2n+1}) = \beta_1 \,;\, \sigma(\beta_1) = \beta_{2n+2} \,;\, \sigma(\beta_{2n+2}) = \beta_o$

$$\alpha_o^\vee = \beta_o^\vee + \beta_1^\vee + \beta_{2n+1}^\vee + \beta_{2n+2}^\vee \qquad e_o = E_o + E_1 + E_{2n+1} + E_{2n+2}$$

$$f_o = F_o + F_1 + F_{2n+1} + F_{2n+2}$$

$$\alpha_n^\vee = \beta_{i+1}^\vee + \beta_{2n+1-i}^\vee \qquad e_i = E_i + E_{2n+1-i} \qquad f_i = F_1 + F_{2n+1-i} \qquad (i=2, \ldots n-1)$$

$$\alpha_n^\vee = \beta_{n+1}^\vee \qquad e_n = E_{n+1} \qquad f_n = F_{n+1}$$

are the Chevalley generators of :

$$g_o^{(o)} = (BC_n \,;\, 1, 1, 0,0,0)$$

The extension leads to :

$$R(\widetilde{g}) \simeq (BC_n \,;\, \nu \,;\, 1, \nu-2, \nu-2, 0)$$

The result of this subsection is an extension of **Theorem 17** to non-reduced gradient root systems :

we get realizations of : $(\mathbf{BC}_n\,;\nu\,;\,1,\nu-1,\nu-1,0)$

$(\mathbf{BC}_n\,;\nu\,;\,1,\nu-2,\nu-2,1)$

$(\mathbf{BC}_n\,;\nu\,;\,2,\nu-2,\nu-2,0)$

$(\mathbf{BC}_n\,;\nu\,;\,1,\nu-1,\nu-2,0)$

$(\mathbf{BC}_n\,;\nu\,;\,1,\nu-2,\nu-2,0)$

In particular, specializing our results to the case $\nu = 2$, and referring to the classification **Theorem 12**, we get this nice conclusion :

Corollary :

All the irreducible elliptic quasisimple root systems of type $\nu = 2$ possess a realization as "genralized loop algebras", or "current algebras", as described in section **III**.

IV. <u>CONCLUSIONS :</u>

The results of section **III** show that at least some of the quasisimple root systems classified in section **II** possess a realization as as current algebra ; unfortunately, due to the lack of a generalized Cartan matrix when $\nu \geqslant 2$, we have not been able to generalize such a construction to twist three QSLA's for instance. Nevertheless, we believe that it is possible to associate a quasisimple Lie algebra to anyone of the rootsystems classified in **Theorem 12** (and perhaps many QSLA's for a given quasisimple root system for $\nu > 2$, as suggested by V.G. Kac) ; abstract constructions are currently under study.

Another very interesting application of the theory of QSLA's is the representation theory, which is probably closely connected to the representation theory of the gauge groups, in quantum gauge theories. Actually, some unitary representations of the gauge group, called the energy representations, have been accurately studied [16], [17], but very little was known about highest weight representations. The quasisimple theory allows the study of such representations [11], [15] ; in particular they exhibit, as a representative of the central extension terms (see section **III**), an expression identical to the Schwinger anomalous term in quantum current algebras [18]. This then appears to be a very promizing field of research, in connection with quantum gauge field theories.

Acknowledgments :

It is a pleasure to thank Pr. S. Albeverio, Pr. V.G. Kac, Pr. M. Sirugue and Pr. D. Testard for fruitful discussions.

REFERENCES

[1] V.G. Kac : "Infinite-dimensional Lie algebras", Cambridge University press (1985).

[2] I.G. Mc Donald : Invent. Math. 15 (1972) p. 91

[3] V.G. Kac : Adv. in Math. 30 (1978) p. 85

[4] Date, Jimbo, Kashiwara, Miwa : J. Phys. Soc. Jap. 50 (1981) p. 3806.

[5] I. Frenkel ; V.G. Kac : Invent. Math. 62 (1980) p.23

[6] J. Mickelsson : Comm. Math. Phys. 97 (1985) p. 361

[7] V.G. Kaç : Math. USSR Izv. 2 (1968) p. 1271

[8] R.V. Moody : J. Alg. 10 (1968) p.211

[9] P. Goddard, D. Olive : MSRI publications 3, Springer Verlag (1984) p.51

[10] J.H. Schwarz : Phys. Rep. 89 (1982) p. 223

[11] B. Torrésani : preprint CPT 86/ P. 1904, to appear in Lett. in Math. Phys.

[12] H. Freudenthal, H. De Vries : "Linear Lie groups". Academic Press (1969)

[13] N. Bourbaki : "Groupes et algèbres de Lie", chap. 4, 5, 6. Elements de mathématiques, Herman

[14] E. Cartan : Thesis (1894)

[15] B. Torrésani : "Representations projectives des groupes de transformations de jauge locales", dissertation, preprint CPT 86/p. 1893

[16] I.M. Gel'fand, M.I. Graev, A.M. Vershik : composition math. 35, 3, (1977), p. 299

[17] S. Albeverio, R. Hoegh-Krohn, D. Testard : J. of functional analysis 41 (1981), p. 378.

[18] I. Bars : MSRI publications 3, Springer Verlag (1984), p. 373.